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SHORT CAPILLARY WAVES ON THE SURFACE OF A STRETCHING CYLINDRICAL JET OF A VISCOUS LIQUID

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This paper presents asymptotic formulas describing the evolution of short-wave perturbations on the surface of a cylindrical viscous liquid jet with the radius decreasing in time. The effects of Reynolds and Weber numbers and the initial wavenumber on the decay of the perturbations are analyzed.

Introduction. The capillary breakup of drops with a nonuniform velocity distribution over the drop length can be explained based on stability analysis for a stretching liquid cylinder under the action of capillary forces. The capillary stability of a stretching cylindrical jet of a viscous liquid immersed in another fluid was studied by Tomotika [1]. The stability of an elongated cylindrical drop in an external flow having a linear velocity distribution over the radius-vector was studied by Khakhar and Ottino [2]. Both [1] and [2] deal with so-called creeping flows, i.e. inertial forces are assumed to be negligibly small compared to viscous friction and surface tension. Papers [3, 4] are devoted to the theoretical analysis of another important case of flows in which inertial forces and surface tension are dominant, while viscous friction and interaction between the jet and the ambient medium can be neglected.

As is shown in [4], after a lapse of sufficiently long time, the greater the initial wavenumber, the higher the growth rate of perturbations. Hence, if at the moment of drop formation the drop surface has initial small perturbations with different wavelengths, secondary breakup of the drop can result from growth of shortest-wave perturbations provided their initial amplitudes are comparable with the amplitudes of the perturbations with longer wavelengths.

In [4], the influence of viscosity on the evolution of perturbations is assumed to be negligibly small. It is well known that for an ideal-liquid jet of constant radius, the presence of short-wave perturbations leads to excitation of harmonic oscillations whose frequency is proportional to the wavenumber to power 3/2. Owing to viscosity, high-frequency oscillations are rapidly damped. In the case of an stretching jet, short-wave perturbations begin to grow rapidly after a lapse of time. The growth of these perturbations can lead to breakup of the jet into drops. Therefore, it is necessary to examine the rate of decay of the perturbation amplitude in the initial period of time. The present paper is devoted to investigation of this problem.

1. Formulation of the Problem. The present paper deals with the capillary stability of a cylindrical jet of an incompressible viscous liquid. The jet radius varies with time. As is shown by Rayleigh, jet breakup into drops results from growth of axisymmetric perturbations. Hence, our attention is focused on this type of perturbations. It is assumed that the liquid velocity is sufficiently high, so that the influence of gravity on the breakup of the liquid jet due to capillary forces can be neglected.

When solving the problem, it is convenient to use nondimensional variables. Let us introduce the following notation: $h(t)$ is the jet radius (t is time), $h_0 = h(0)$ is the initial jet radius, ρ and μ are the density and viscosity of the liquid, and σ is the surface tension at the interface between the liquid jet and the ambient medium (gas). Then, h_0, $t_0 = -h_0/(2h_t(0))$, h_0/t_0, and $p_0 = \sigma/h_0$ can be used as the length, time, velocity, and pressure scales, respectively. Here $h_t(0)$ denotes the time derivative of the jet radius at $t = 0$. We introduce a cylindrical coordinate system (r, θ, z), where the z axis coincides with the symmetry axis. The axial and radial components of the liquid-velocity vector are denoted by u and v, respectively. The liquid pressure is denoted by p. The governing equations

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for liquid motion in the jet includes the continuity equation

$$v_r + v/r + u_z = 0, \tag{1.1}$$

the axial component of the Navier–Stokes equation

$$u_t + vu_r + uu_z = -\frac{1}{\mathrm{We}}\, p_z + \frac{1}{\mathrm{Re}}\Big[\frac{1}{r}\,(ru_r)_r + u_{zz}\Big], \tag{1.2}$$

and the radial component of the Navier–Stokes equation

$$v_t + vv_r + uv_z = -\frac{1}{\mathrm{We}}\, p_r + \frac{1}{\mathrm{Re}}\Big[\frac{1}{r}\,(rv_r)_r - \frac{v}{r^2} + v_{zz}\Big]. \tag{1.3}$$

Hereinafter, the subscripts denote derivatives with respect to corresponding variables, i.e., $v_r = \partial v/\partial r$ and so forth. The Reynolds and Weber numbers are defined by $\mathrm{Re} = \rho h_0^2/(\mu t_0^2)$ and $\mathrm{We} = \rho h_0^3/(\sigma t_0^2)$. On the free surface of the jet described by the equation $r = h(z,t)$, we impose the kinematic boundary condition

$$h_t + uh_z = v \quad \text{at} \quad r = h \tag{1.4}$$

and two dynamic boundary conditions. The first condition requires that the tangential component of the viscous stress be zero:

$$2v_r h_z + (u_r + v_z)(1 - h_z^2) - 2u_z h_z = 0 \quad \text{for} \quad r = h. \tag{1.5}$$

According to the second condition, the normal components of the stress vector on the inner and outer sides of the jet surface differ by the value $\sigma(1/R_1 + 1/R_2)$, where R_1 and R_2 are the principal radii of the jet surface curvature in the longitudinal and transverse directions. In the nondimensional variables, this condition is written as

$$p - \frac{\mathrm{We}}{\mathrm{Re}}\,[2v_r - (u_r + v_z)h_z] = p_a + \frac{1}{R_1} + \frac{1}{R_2}. \tag{1.6}$$

Here p_a is the ambient pressure, which is considered constant. The principal radii of curvature in the cylindrical coordinate system are expressed as

$$\frac{1}{R_1} = \frac{1}{h(1+h_z^2)^{1/2}}, \qquad \frac{1}{R_2} = -\frac{h_{zz}}{(1+h_z^2)^{3/2}}.$$

The problem formulated above allows the following representation of the solution. We assume that the jet shape remains unchanged during deformation, i.e., $h_z = 0$. The continuity equation (1.1) and boundary conditions (1.4) and (1.5) are satisfied when the velocity components have the form

$$u = u_1 = z/\tau, \qquad v = v_1 = -r/(2\tau), \tag{1.7}$$

where $\tau = 1 + t$. The equations of liquid motion (1.2) and (1.3) and boundary condition (1.6) are satisfied if the jet radius varies with time as

$$h = h_1 = \tau^{-1/2}, \tag{1.8}$$

and the liquid pressure is given by

$$p = p_1 = p_a + \tau^{1/2} + \frac{3\,\mathrm{We}}{8\tau^3}(1 - \tau r^2) - \frac{\mathrm{We}}{\mathrm{Re}\,\tau}. \tag{1.9}$$

The above formulas differ from those presented in [4] by the presence of an additional term in the expression for the liquid pressure.

2. Equations for Small Perturbations of the Main Flow. Expressions (1.7)–(1.9) describe the uniaxial extension of the liquid jet with undistorted jet shape. Let us assume that at the initial time the main flow is subjected to small perturbations:

$$u = u_1 + \delta u, \quad v = v_1 + \delta v, \quad h = h_1 + \delta h, \quad p = p_1 + \delta p. \tag{2.1}$$

Here δu, δv, δh, and δp are perturbations of the velocity components, the jet radius, and the liquid pressure, which are assumed to be small compared to u_1, v_1, h_1, and p_1, respectively.

The equations of small perturbations of the main flow and the corresponding boundary conditions can easily be obtained by substituting (2.1) into (1.1)–(1.6) and omitting the terms containing the products of small quantities. Let us introduce the stream function $\delta\psi$: $r\delta v = \delta\psi_z$, $r\delta u = -\delta\psi_r$, and the quantity $\delta\zeta$, which is proportional to the vorticity of the velocity field $[\delta\zeta = r(\delta v_z - \delta u_r)]$. The equation for the stream function $\delta\psi$ follows from the linearized continuity equation, and the equation for $\delta\zeta$ is derived from the linearized equations of liquid motion by eliminating δp. We assume that at $t = 0$ (i.e., $\tau = 1$), the initial perturbation of the jet radius is sinuous. This initial condition can be satisfied if the quantities δh, $\delta\psi$, and $\delta\zeta$ are sought in the form

$$\delta h = i\tau^{-1/2}\eta(t)\exp(ikz), \quad \delta\psi = r\tau^{-1/2}f(r,t)\exp(ikz), \quad \delta\zeta = r\tau^{-1/2}g(r,t)\exp(ikz),$$

where i is imaginary unit, $k(t)$ is the wavenumber, and the factors $\tau^{-1/2}$ and $r\tau^{-1/2}$ are introduced for convenience. The variable z is eliminated from the equations if the wavenumber varies with time as $k = k_0/\tau$, where $k_0 = k(0)$ is the initial value of the wavenumber. Instead of r we introduce the new independent variable $\xi = r\tau^{1/2}$. The equations for $\delta\psi$ and $\delta\zeta$ lead to the following equations for the unknown quantities f and g:

$$g_{\xi\xi} - \Big(\frac{3}{4\xi^2} + \frac{k^2}{\tau}\Big)g = \frac{\mathrm{Re}}{\tau}\, g_t; \tag{2.2}$$

$$f_{\xi\xi} - \Big(\frac{3}{4\xi^2} + \frac{k^2}{\tau}\Big)f = \frac{g}{\tau}. \tag{2.3}$$

In the new variables, the boundary conditions on the jet surface ($\xi = 1$) are

$$\eta_t = kf, \tag{2.4}$$

$$2k^2 f + g = 3k\eta/\tau, \tag{2.5}$$

$$2k^2 f_\xi - g_\xi + \mathrm{Re}\Big(f_{\xi t} + \frac{1}{2}\, f_t + \frac{f_\xi}{\tau}\Big) = \frac{3k\eta}{2\tau} + \frac{\mathrm{Re}}{\mathrm{We}}(k\tau^{1/2} - k^3\tau^{-1/2})\eta - \frac{3\mathrm{Re}}{8}\Big(2k\tau^{-3}\eta + \frac{\eta_t}{k\tau}\Big). \tag{2.6}$$

3. Short-Wave Asymptotic Relations. To study the decay of short-wave perturbations over the initial period of time, we derive short-wave asymptotic formulas assuming that the initial value of the wavenumber k_0 is large ($k_0 \to \infty$). As for an ideal liquid [4], such asymptotic formulas cannot be uniformly valid for any t. In deriving the asymptotic formulas, we assume that time t is finite. Since perturbations with small wavelength compared to the jet radius are little different from capillary waves on a flat surface, we seek a solution in the form

$$\eta = \exp(\omega_0(\tau)k_0^2 + \omega_1(\tau)k_0)(\eta_0 + k_0^{-1}\eta_1 + \ldots); \tag{3.1}$$

$$g = k_0^3\exp(\omega_0(\tau)k_0^2 + \lambda_1(\xi,\tau)k_0)(g_0 + k_0^{-1}g_1 + \ldots). \tag{3.2}$$

The unknown function f is represented as the sum $f = f_1 + f_2$, where f_1 is a particular solution of (2.3) and f_2 is the general solution of the corresponding homogeneous equation; in addition,

$$f_i = k_0\exp(\omega_0(\tau)k_0^2 + \lambda_i(\xi,\tau)k_0)(f_{i0} + k_0^{-1}f_{i1} + \ldots) \quad (i = 1, 2). \tag{3.3}$$

The factors k_0^3 and k_0 on the right sides of Eqs. (3.2) and (3.3) are introduced since the right and left sides of Eqs. (2.3) and (2.4) should have the same order of magnitude for large k_0. Substituting (3.2) into (2.2) and equating the coefficients at the same powers of k_0, we obtain

$$\lambda_{1\xi}^2 - \tau^{-3} = \frac{\mathrm{Re}\,\omega_{0t}}{\tau},$$
$$2\lambda_{1\xi}g_{0\xi} = \frac{\mathrm{Re}\,\lambda_{1t}g_0}{\tau}, \qquad 2\lambda_{1\xi}g_{1\xi} + g_{0\xi\xi} - \frac{3g_0}{4\xi^2} = \frac{\mathrm{Re}}{\tau}\,(\lambda_{1t}g_1 + g_{0t}). \tag{3.4}$$

As follows from (3.4), the value of $\lambda_{1\xi}$ does not depend on ξ, i.e., λ_1 is a linear function of ξ. Taking into account (3.4), from the nonhomogeneous equation (2.3) we obtain the relations

$$\mathrm{Re}\,\omega_{0t}f_{10} = g_0, \qquad \mathrm{Re}\,\omega_{0t}f_{11} + \mathrm{Re}\,\lambda_{1t}f_{10} = g_1,$$
$$\mathrm{Re}\,\omega_{0t}f_{12} + 2\lambda_{1\xi}\tau f_{11\xi} + \tau\Big(f_{10\xi\xi} - \frac{3f_{10}}{4\xi^2}\Big) = g_2. \tag{3.5}$$

The homogeneous equation corresponding to (2.3) yields

$$\lambda_{2\xi}^2 = \tau^{-3}, \qquad f_{20\xi} = 0, \qquad f_{21\xi} = \frac{3f_{20}}{8\xi^2\lambda_{2\xi}}. \tag{3.6}$$

We consider the boundary conditions on the jet surface. Since the exponents on the right sides of (3.1)–(3.3) should coincide on the jet surface ($\xi = 1$), we have $\lambda_1 = \lambda_{1\xi}(\xi - 1) + \omega_1$ and $\lambda_2 = \lambda_{2\xi}(\xi - 1) + \omega_1$. Boundary conditions (2.4) and (2.5) require that the following identities be satisfied at $\xi = 1$:

$$\omega_{0t}\eta_0 = \frac{1}{\tau}(f_{10} + f_{20}), \quad \omega_{0t}\eta_1 + \omega_{1t}\eta_0 = \frac{1}{\tau}(f_{11} + f_{21}),$$

$$\omega_{0t}\eta_2 + \omega_{1t}\eta_1 + \eta_{0t} = \frac{1}{\tau}(f_{12} + f_{22}), \tag{3.7}$$

$$\frac{2}{\tau^2}(f_{10} + f_{20}) + g_0 = 0, \quad \frac{2}{\tau^2}(f_{11} + f_{21}) + g_1 = 0, \quad \frac{2}{\tau^2}(f_{12} + f_{22}) + g_2 = \frac{3}{\tau^2}\eta_0.$$

We consider boundary condition (2.6). For large k_0, the leading terms on the right side of (2.6) are proportional to k_0^4/Re; on the left side of (2.6), the leading term is proportional to $k_0^3\,\mathrm{Re}/\mathrm{We}$. The ratio of these coefficients is $\mathrm{Re}^2/(\mathrm{We}\,k_0) = (\rho\sigma/\mu^2)(h_0/k_0)$. It is worth noting that for physically meaningful perturbations, this parameter is not small despite the presence of the large quantity k_0 in the denominator. Indeed, the value of h_0/k_0 is 2π times larger than the perturbation wavelength. Consideration of perturbations for which this parameter is smaller than 10^{-5} m is physically meaningless. At the same time, the parameter $\rho\sigma/\mu^2$ is of order 10^7 m^{-1} for usual low-viscosity liquids similar to water in their physical properties. Therefore, the parameter $\mathrm{Re}^2/(\mathrm{We}\,k_0)$ is large rather than small. This consideration can be used to derive approximate formulas. Let us introduce the parameter $\varepsilon = \mathrm{We}\,k_0/\mathrm{Re}^2$, which is assumed to have a finite value as $\mathrm{Re} \to \infty$. Then, equating higher-order terms on the left and right sides of (2.6) and rearranging the obtained relations by using (3.5)–(3.7), we obtain the following algebraic equation for $\lambda_{1\xi}$:

$$\varepsilon[(1 + \tau^3\lambda_{1\xi}^2)^2 - 4\tau^{3/2}\lambda_{1\xi}] = -\tau. \tag{3.8}$$

Combining (3.8) with the ordinary differential equation (3.4), we obtain the dependence $\omega_0(t)$ for ω_0. An approximate solution can readily be obtained assuming that ε is a small parameter. Let us introduce the new unknown variable y so that $\tau^{3/2}\lambda_{1\xi} = \varepsilon^{-1/4}y$ and express y as the infinite series $y = y_0 + \varepsilon^{1/2}y_1 + \varepsilon^{3/4}y_2 + \varepsilon y_3 + \dots$. The term of order $\varepsilon^{1/4}$ is omitted since (3.8) expressed in terms of y does not contain a term proportional to y^3. As a result, we obtain

$$y = y_0 - \frac{1}{2y_0}\varepsilon^{1/2} + \frac{1}{y_0^2}\varepsilon^{3/4} - \frac{1}{8y_0^3}\varepsilon + \dots,$$

where y_0 is defined by the equation $y_0^4 + \tau = 0$. Then, using (3.4), we have

$$\omega_{0t} = \frac{1}{\mathrm{Re}\,\tau^2}\Big(y_0^2\varepsilon^{-1/2} - 2 + \frac{2}{y_0^2}\varepsilon^{1/4} - \frac{1}{y_0^3}\varepsilon^{3/4} + \dots\Big).$$

Integration of this ordinary differential equation yields the following expression for ω_0:

$$\omega_0 = \pm\frac{2i\tau^{-1/2}}{\mathrm{We}^{1/2}k_0^{1/2}} + \frac{2\tau^{-1}}{\mathrm{Re}} + \dots\ .$$

Let us determine the quantity ω_1. The equation for ω_1 is derived by equating the coefficients at k_0^3 on the left and right sides of (2.6) and rearranging the expression obtained:

$$\omega_{1t}\Big(1 + \tau^3\lambda_{1\xi}^2 - \frac{1}{\tau^{3/2}\lambda_{1\xi}}\Big) = -\frac{1}{4}\tau^{7/2}\mathrm{Re}\,\omega_{0t}^2.$$

For small ε, the approximate solution of this equation has the form

$$\omega_1 = \pm\frac{i\tau}{4\,\mathrm{We}^{1/2}k_0^{1/2}} + \frac{2\tau^{1/2}}{\mathrm{Re}} + \dots\ .$$

Since the differential equation for $\eta_0(t)$ is rather cumbersome, we give the following approximate expression for small ε:

$$\eta_{0t} = \eta_0\Big(\mp\frac{19}{32}\,i\,\frac{\tau^{3/2}}{\mathrm{We}^{1/2}k_0^{1/2}} - \frac{\tau}{4\,\mathrm{Re}} + \frac{1}{2\tau} + \dots\Big).$$

Thus, the evolution of perturbations of the stretching jet radius for large initial wavenumber k_0 is described by the following asymptotic formula:

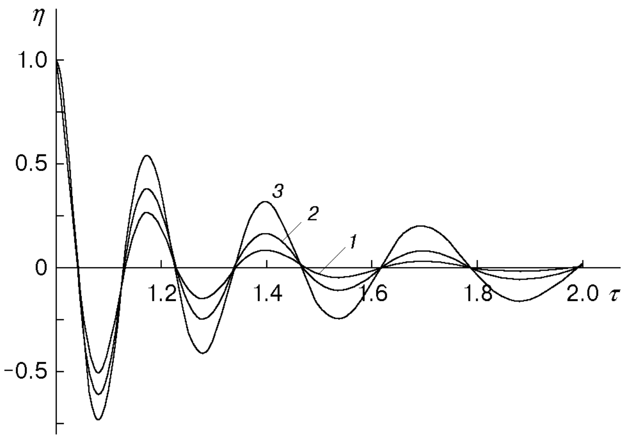

Fig. 1. Perturbation of jet radius η versus time τ at $k_0 = 12$, $\mathrm{We} = 1$, and $\mathrm{Re} = 30$ (1), 40 (2), and 60 (3).

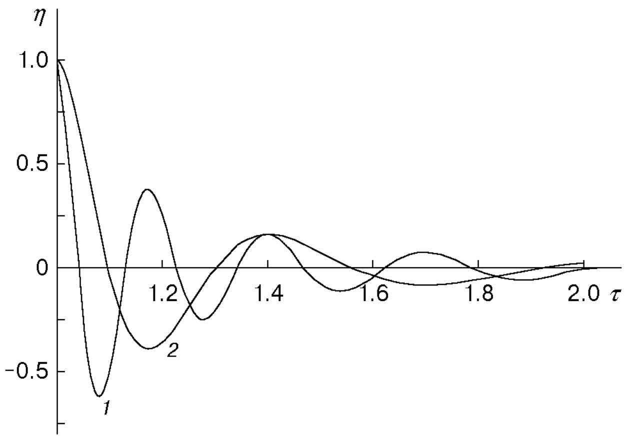

Fig. 2

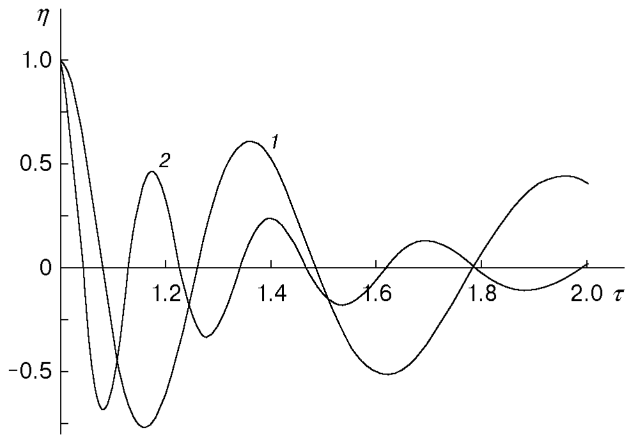

Fig. 3

Fig. 2. Perturbation of the jet radius η versus time τ at $k_0 = 12$, $\mathrm{Re} = 40$, and $\mathrm{We} = 1$ (1) and 2 (2).

Fig. 3. Perturbation of the jet radius η versus time τ at $\mathrm{Re} = 50$, $\mathrm{We} = 1$, and $k_0 = 8$ (1) and 12 (2).

$$\eta = \tau^{1/2} \exp\Big(\frac{1}{\mathrm{Re}}\Big(\frac{2k_0^2}{\tau} + 2k_0\tau^{1/2} - \frac{\tau^2}{8}\Big)\Big)(A\cos\alpha(t) + B\sin\alpha(t)). \qquad (3.9)$$

Here A and B are the constants of integration and the quantity $\alpha(t)$ is given by

$$\alpha(t) = \frac{1}{\mathrm{We}^{1/2}}\Big(\frac{2k_0^{3/2}}{\tau^{1/2}} + \frac{k_0^{1/2}\tau}{4} + \frac{19\tau^{5/2}}{80k_0^{1/2}}\Big).$$

It should be noted that for $\mathrm{Re} \to \infty$, the expressions obtained agree with the asymptotic formulas presented in [4]. Time dependences of the jet radius perturbation are shown in Figs. 1–3. The dependence shown in Fig. 1 is calculated for the initial conditions $\eta(0) = 1$ and $\eta_t(0) = 0$ and different values of Re. Viscosity causes the oscillation amplitude to decay. The lower Re, the higher the rate of decay. However, this decay is slower than the decay of perturbations with the same wavelength in the case of a jet with constant radius. If the value of Re is not too small, the at the time when the beginning of perturbation growth is expected, the amplitude calculated from (3.9) is also not small. For example, at $\tau = 6$, $k_0 = 12$, $\mathrm{We} = 1$, and $\mathrm{Re} = 40$, the perturbation amplitude calculated from (3.9) for the above-mentioned initial conditions is $\eta = 0.0128$. As follows from Fig. 2, the oscillation frequency decreases with increasing We. As the initial wavenumber decreases, the decay rate of the perturbations also decreases (Fig. 3). The asymptotic formulas derived in the present paper are not appropriate for large τ. In this case, a solution can be derived using approximate one-dimensional equations for the dynamics of a capillary jet.

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